Journal of Pure and Applied Algebra 36 (1985) 219–220
North-Holland

NOTE

LOCALLY FINITE ADEQUATE SUBCATEGORIES

John ISBELL
State University of New York at Buffalo, 106 Diefendorf Hall, Buffalo, NY 14214, USA

Communicated by F.W. Lawvere
Received August 1984

This note shows that a complete (or countably complete) category $\mathscr{C}$ which is not preordered cannot have a two-sided adequate subcategory all of whose hom sets are finite. The cardinal $\aleph_0$ can be replaced by any other measurable cardinal m (that is, having a non-trivial 2-valued measure that is n-additive for all $n<m$). Categories of vector spaces show that such a result holds only for measurable cardinals [2]. By the way, there are also arbitrarily large categories in which a seven-element monoid is two-sided adequate [3].

The proof involves compact Hausdorff topologies, or more generally what we may call m-small topologies: every ultrafilter that is n-multiplicative for all $n<m$ has a unique limit point. Also we get a stronger result and a more transparent proof by decomposing adequacy. Recall that a subcategory $\mathscr{B}$ is right adequate *for* an object X if morphisms $T\to X$ (for all T in the category) correspond bijectively to natural transformations from $\mathrm{Hom}(X,\cdot)\,|\,\mathscr{B}$ to $\mathrm{Hom}(T,\cdot)\,|\,\mathscr{B}$ [1]. The equivalence between global right adequacy and every object X being the limit of the canonical diagram on objects indexed by $\bigcup[\mathrm{Hom}(X,B)\colon B\in|\mathscr{B}|]$ (the f-th object being the codomain of f, the f-th and g-th objects joined by the morphisms $h\in\mathscr{B}$ for which $hf=g$) [4, X. 6.2] localizes to single objects X. We can also remove a redundant assumption by recalling [1, 9.5.a] that if $\mathscr{B}$ is right adequate for certain objects, notably X^m, which are right adequate for another object X (any object is adequate for its retracts), and every morphism $X\to B$ factors through X^m, then $\mathscr{B}$ is right adequate for X. (In [1] $\mathscr{B}$ is full, but that doesn't enter in the proof.)

Theorem. *Let m_0 be a measurable cardinal, and let $\mathscr{C}$ be a category having a left adequate subcategory $\mathscr{A}$ and a subcategory $\mathscr{B}$ such that* $\mathrm{Hom}(A,B)$ *has less than m_0 elements for all A in $\mathscr{A}$, B in $\mathscr{B}$. Let X be an object of $\mathscr{C}$ and m a cardinal not less than m_0 such that X has an m-th power X^m in $\mathscr{C}$ and $\mathscr{B}$ is right adequate for X^m. Then X^m is X, i.e., every hom set* $\mathrm{Hom}(C,X)$ *has at most one element.*

Proof. The effect of the assumptions involving $\mathscr{A}$ is that $\mathscr{C}$ is fully embedded in $\mathrm{Cat}(\mathscr{A}^{\mathrm{op}},\mathscr{S})$ preserving limits, and each B embeds as an m_0-small-set-valued functor $\tilde{B}$. Also, to show that every Hom(C, X) has at most one element we need only prove it for sets Hom$(A, X) = \tilde{X}(A)$, A in $\mathscr{A}$.

When $\mathscr{B}$ is right adequate for Y (as it is for X as well as X^m, as we noted), the sets $\tilde{Y}(A)$ bear m_0-small topologies as limits of the small sets $\tilde{B}(A)$ with discrete topology; every natural transformation $\varphi : \tilde{Y} \to \tilde{B}$ (B in $\mathscr{B}$) has continuous components φ_A. Also, $\tilde{Y}$ takes morphisms $A \to A'$ to continuous functions $\tilde{Y}(A') \to \tilde{Y}(A)$. Let p be an ultrafilter on the index set m for the product X^m which is n-multiplicative for $n < m_0$. Then for each A in $\mathscr{A}$, each m-tuple $\{x_\alpha]$ in $\tilde{X}^m(A)$ converges along p, in the space $\tilde{X}(A)$, to a unique limit point $\Pi_A(\{x_\alpha\})$. Since $\tilde{X}$ is continuous-valued, the functions Π_A are the components of a natural transformation $\Pi : \tilde{X}^m \to \tilde{X}$. But if some $\tilde{X}(A)$ had two different elements u, v, Π_A would take all points that have only finitely many coordinates different from u to u and their limit point all of whose coordinates are v to v.

References

[1] J. Isbell, Uniform neighborhood retracts, Pacific J. Math. 11 (1961) 609–648.

[2] J. Isbell, Subobjects, adequacy, completeness and categories of algebras, Rozprawy Mat. 36 (1964).

[3] J. Isbell, Small adequate subcategories, J. London Math. Soc. 43 (1968) 242–246.

[4] S. MacLane, Categories for the Working Mathematician (Springer, New York, 1971).